$gl_q(n)$-covariant Oscillators and q-Deformed Quantum Mechanics in n Dimensions

W.-S.Chung

Theory Group, Department of Physics, College of Natural Sciences,
Gyeongsang National University,
Jinju, 660-701, Korea

Abstract

In this paper the coherent state for $gl_q(n)$-covariant oscillators is constructed and is shown to satisfy the completeness relation. And the q-analogue of quantum mechanics in n dimensions is obtained by using $gl_q(n)$ oscillators.

1 Introduction

Quantum groups or q-deformed Lie algebra implies some specific deformations of classical Lie algebras.

From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6-10].

In the mean time some theoretical physicists studied the q-deformation of quantum mechanic in one dimension [11-16]. The purpose of this paper is to use the $gl_q(n)$-covariant oscillator algebra to construct the q-analogue of the quantum mechanics with harmonic potential in n dimensions.

2 Coherent states of $gl_q(n)$-covariant oscillator algebra

$gl_q(n)$-covariant oscillator algebra is defined as [17]

$$a_i^\dagger a_j^\dagger = \sqrt{q} a_j^\dagger a_i^\dagger, \quad (i < j)$$

$$a_i a_j = \frac{1}{\sqrt{q}} a_j a_i, \quad (i < j)$$

$$a_i a_j^\dagger = \sqrt{q} a_j^\dagger a_i, \quad (i \neq j)$$

$$a_i a_i^\dagger = 1 + q a_i^\dagger a_i + (q-1)\Sigma_{k=i+1}^n a_k^\dagger a_k, \quad (i = 1, 2, \cdots, n-1)$$

$$a_n a_n^\dagger = 1 + q a_n^\dagger a_n,$$

$$[N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \quad (i, j = 1, 2, \cdots, n) \tag{1}$$

where we restrict our concern to the case that q is real and $0 < q < 1$. Here N_i plays a role of number operator and $a_i(a_i^\dagger)$ plays a role of annihilation(creation) operator. From the above algebra one can obtain the relation between the number operators and mode opeartors as follows

$$a_i^\dagger a_i = q^{\Sigma_{k=i+1}^n N_k}[N_i], \tag{2}$$

where $[x]$ is called a q-number and is defined as

$$[x] = \frac{q^x - 1}{q - 1}.$$

Let us introduce the Fock space basis $|n_1, n_2, \cdots, n_n >$ for the number operators $N_1, N_2, \cdots, N_n$ satisfying

$$N_i|n_1, n_2, \cdots, n_n >= n_i|n_1, n_2, \cdots, n_n >, \quad (n_1, n_2, \cdots, n_n = 0, 1, 2 \cdots) \quad (3)$$

Then we have the following representation

$$a_i|n_1, n_2, \cdots, n_n >= \sqrt{q^{\Sigma_{k=i+1}^n n_k}[n_i]}|n_1, \cdots, n_i - 1, \cdots, n_n >$$

$$a_i^\dagger|n_1, n_2, \cdots, n_n >= \sqrt{q^{\Sigma_{k=i+1}^n n_k}[n_i + 1]}|n_1, \cdots, n_i + 1, \cdots, n_n > . \quad (4)$$

From the above representation we know that there exists the ground state $|0, 0, \cdots, 0 >$ satisfying $a_i|0, 0 >= 0$ for all $i = 1, 2, \cdots, n$. Thus the state $|n_1, n_2, \cdots, n_n >$ is obtatind by applying the creation operators to the ground state $|0, 0, \cdots, 0 >$

$$|n_1, n_2, \cdots, n_n >= \frac{(a_n^\dagger)^{n_n} \cdots (a_1^\dagger)^{n_1}}{\sqrt{[n_1]! \cdots [n_n]!}}|0, 0, \cdots, 0 > . \quad (5)$$

If we introduce the scale operators as follows

$$Q_i = q^{N_i}, \quad (i = 1, 2, \cdots, n), \quad (6)$$

we have from the algebra (1)

$$[a_i, a_i^\dagger] = Q_i Q_{i+1} \cdots Q_n. \quad (7)$$

Acting the operators Q_i's on the basis $|n_1, n_2, \cdots, n_n>$ produces

$$Q_i|n_1, n_2, \cdots, n_n >= q^{n_i}|n_1, n_2, \cdots, n_n > . \tag{8}$$

From the relation $a_i a_j = \frac{1}{\sqrt{q}} a_j a_i, \quad (i < j)$, the coherent states for $gl_q(n)$ algebra is defined as

$$a_i|z_1, \cdots, z_i, \cdots, z_n >= z_i|z_1, \cdots, z_i, \sqrt{q} z_{i+1}, \cdots, \sqrt{q} z_n > . \tag{9}$$

Solving the eq.(9) we obtain

$$|z_1, z_2, \cdots, z_n >= c(z_1, \cdots, z_n)\Sigma_{n_1,n_2,\cdots,n_n=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} \cdots z_n^{n_n}}{\sqrt{[n_1]![n_2]! \cdots [n_n]!}}|n_1, n_2, \cdots, n_n > . \tag{10}$$

Using eq.(5) we can rewrite eq.(10) as

$$|z_1, z_2, \cdots, z_n >= c(z_1, \cdots, z_n) \exp_q(z_n a_n^\dagger) \cdots \exp_q(z_2 a_2^\dagger) \exp_q(z_1 a_1^\dagger)|0, 0, \cdots, 0 > . \tag{11}$$

where q-exponential function is defined as

$$\exp_q(x) = \Sigma_{n=0}^{\infty} \frac{x^n}{[n]!}.$$

The q-exponential function satisfies the following recurrence relation

$$\exp_q(qx) = [1 - (1 - q)x] \exp_q(x) \tag{12}$$

Using the above relation and the fact that $0 < q < 1$, we obtain the formula

$$\exp_q(x) = \Pi_{n=0}^{\infty} \frac{1}{1 - (1 - q)q^n x} \tag{13}$$

Using the normalization of the coherent state , we have

$$c(z_1, z_2, \cdots, z_n) = \exp_q(|z_1|^2)\exp_q(|z_2|^2)\cdots\exp_q(|z_n|^2). \quad (14)$$

The coherent state satisfies the completeness relation

$$\int\cdots\int |z_1, z_2, \cdots, z_n >< z_1, z_2, \cdots, z_n|\mu(z_1, z_2, \cdots, z_n)d^2z_1d^2z_2\cdots d^2z_n = I, \quad (15)$$

where the weighting function $\mu(z_1, z_2, \cdots, z_n)$ is defined as

$$\mu(z_1, z_2, \cdots, z_n) = \frac{1}{\pi^2}\Pi_{i=1}^n\frac{\exp_q(|z_i|^2)}{\exp_q(q|z_i|^2)}. \quad (16)$$

In deriving eq.(15) we used the formula

$$\int_0^{1/(1-q)} x^n \exp_q(qx)^{-1}d_qx = [n]! \quad (17)$$

3 q-Deformed Weyl-Heisenberg Group

The purpose of this section is to explain what is the q-analogue of the q-deformed Weyl-Heisenberg group. From the algebra (1) we obtain

$$a_if(a_i^\dagger) = f(qa_i^\dagger)a_i + (Df)(a_i^\dagger)Q_{i+1}\cdots Q_n$$

$$a_nf(a_n^\dagger) = f(qa_n^\dagger)a_n + (Df)(a_n^\dagger), \quad (18)$$

where D is called q-derivative and defined as

$$DF(x) = \frac{F(x) - F(qx)}{x(1-q)}.$$

Putting $f(x) = \exp_q(tx)$ we have

$$a_i \exp_q(ta_i^\dagger) = \exp_q(qta_i^\dagger)a_i + t\exp_q(ta_i^\dagger)Q_{i+1}\cdots Q_n. \tag{19}$$

Using the formula (12) we have

$$a_i^n \exp_q(ta_i^\dagger) = \exp_q(ta_i^\dagger)(a_i + tQ_iQ_{i+1}\cdots Q_n)^n \tag{20}$$

and thus

$$\exp_q(s_ia_i)\exp_q(t_ia_i^\dagger) = \exp_q(t_ia_i^\dagger)\exp_q(s_ia_i + s_it_iQ_iQ_{i+1}\cdots Q_n). \tag{21}$$

Taking account of $[a_i, Q_i]_q = a_iQ_i - qQ_ia_i = 0$,we have

$$\exp_q(s_ia_i)\exp_q(t_ia_i^\dagger) = \exp_q(t_ia_i^\dagger)\exp_q(s_it_iQ_iQ_{i+1}\cdots Q_n)\exp_q(s_ia_i). \tag{22}$$

If we muliply above equations from $i = 1$ to n, we obtain the q-deformed Weyl-Heisenberg relation

$$\Pi_{i=1}^n \exp_q(s_ia_i)\exp_q(t_ia_i^\dagger) = \Pi_{i=1}^n \exp_q(t_ia_i^\dagger)\exp_q(s_it_iQ_iQ_{i+1}\cdots Q_n)\exp_q(s_ia_i). \tag{23}$$

4 q-deformed quantum mechanics in n dimensions

It is intersting to study the q-deformed harmonic oscillator system in n dimensions. In order to formulate it we define the position and momentum

operators

$$X_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger)$$

$$P_i = -\frac{i}{\sqrt{2}}(a_i - a_i^\dagger). \tag{24}$$

Then the Hamiltonian of q-deformed harmonic oscillator in n dimensions is given by

$$H = \Sigma_{i=1}^n H_i, \tag{25}$$

where

$$H_i = \frac{1}{2}(P_i^2 + X_i^2) = \frac{1}{2}(a_i a_i^\dagger + a_i^\dagger a_i). \tag{26}$$

Now, the q-cannonical commutation relation can be expressed by

$$X_i P_i - P_i X_i = i(\frac{q+1}{2})^{i-n-1} + i(q-1)\Sigma_{k=i}^n (\frac{q+1}{2})^{i-k-1} H_k. \tag{27}$$

Expressing H_i's in terms of Q_i's operators, we get

$$H_i = \frac{q+1}{2(q-1)} Q_i Q_{i+1} \cdots Q_n - \frac{1}{q-1} Q_{i+1} Q_{i+2} \cdots Q_n, \quad (i = 1, 2, \cdots, n-1)$$

$$H_n = \frac{q+1}{2(q-1)} Q_n - \frac{1}{q-1}. \tag{28}$$

Thus the Hamiltonian is given by

$$H = \frac{Q-1}{q-1} + \frac{1}{2}\Sigma_{i=1}^n Q_i Q_{i+1} \cdots Q_n \tag{29}$$

where

$$Q = Q_1 Q_2 \cdots Q_n$$

Thus we have

$$H|n_1, \cdots, n_n >= E(n_1, \cdots, n_n)|n_1, \cdots, n_n > \tag{30}$$

where the energy spectrum is given by

$$E(n_1, \cdots, n_2) = [n_1 + \cdots + n_n] + \frac{1}{2}\Sigma_{i=1}^{n} q^{n_1+\cdots+n_n} \tag{31}$$

5 Concluding Remark

In this paper we used $gl_q(n)$-covariant oscillator algebra to obtain its coherent state and showed the completeness relation. Moreover we construct the q-deformed quantum mechanical hamiltonian in n dimensions by using $gl_q(n)$-covariant oscillators. In conclusion, it was known that we can obtain the q-analogue of n-dimensional Schroedinger equation with harmonic potential by using $gl_q(n)$-covariant oscillator system.

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